

The bilateral trade model in a discrete setting

Citation for published version (APA):

Flesch, J., Schröder, M. J. W., & Vermeulen, A. J. (2013). *The bilateral trade model in a discrete setting*. Maastricht University, Graduate School of Business and Economics. GSBE Research Memoranda No. 025 <https://doi.org/10.26481/umagsb.2013025>

Document status and date:

Published: 01/01/2013

DOI:

[10.26481/umagsb.2013025](https://doi.org/10.26481/umagsb.2013025)

Document Version:

Publisher's PDF, also known as Version of record

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.umlib.nl/taverne-license

Take down policy

If you believe that this document breaches copyright please contact us at:

repository@maastrichtuniversity.nl

providing details and we will investigate your claim.

János Flesch, Marc Schröder,
Dries Vermeulen

**The bilateral trade model in a
discrete setting**

RM/13/025

GSBE

Maastricht University School of Business and Economics
Graduate School of Business and Economics

P.O. Box 616
NL- 6200 MD Maastricht
The Netherlands

The bilateral trade model in a discrete setting

János Flesch*, Marc Schröder*, Dries Vermeulen*

February 27, 2013

Abstract

We consider a bilateral trade model in which both players have a finite number of possible valuations. The seller's valuation and the buyer's valuation for the object are private information, but the independent beliefs about these valuations are common knowledge. In this setting, we provide a characterization of the set of interim individually rational-implementable trading rules, analogous to the result of Myerson and Satterthwaite [1983]. Thereafter, we derive necessary conditions for incentive compatible and ex post individually rational direct mechanisms. For the special class of corner mechanisms with discrete uniform beliefs, we characterize the set of ex post individually rational-implementable trading rules. In this context it is also shown that ex post efficiency can only be achieved if the number of different valuations is small. The maximal number of different valuations for which efficiency is still possible depends on the prior probability distribution of valuations.

1 Introduction

A fundamental problem in economics is the bilateral trade problem. Distinctive for bilateral trade is that we have just one seller who wants to trade one single indivisible object with only one potential buyer. The main issue with this problem is asymmetric information; each trader's own valuation is private information, and thus there are incentives to misreport this private information. Myerson and Satterthwaite [1983] showed that if we want both players to report truthfully, to be always willing to participate and to have no outside party who is willing to subsidize, then there is no bargaining mechanism that is ex post efficient. One of their assumptions however is that we have a continuum of types for each player. There is however no modeling reason to prefer a continuous type space to a discrete one [Vohra, 2011]. In some applications, people reason with a limited amount of possible valuations. Think, for example, about monetary transactions. We will therefore consider a bilateral trade model in which each player has a finite number of possible valuations. This enables us to derive a more intuitive explanation of some of the classical results on the bilateral trade model.

One other drawback in the analysis of Myerson and Satterthwaite [1983] is that they only consider interim individual rationality, meaning that each player's expected utility conditional on his type needs to be nonnegative. This does not rule out the situation in which, for example, a certain type of buyer is required to make a payment without trade taking place. In most markets

*Address: Department of Quantitative Economics, Maastricht University, P.O. Box 616, 6200 MD Maastricht, The Netherlands. Email: m.schroeder@maastrichtuniversity.nl, j.flesch@maastrichtuniversity.nl, d.vermeulen@maastrichtuniversity.nl

this does not occur, since ex post enforcement of payment is usually not feasible when the terms of trade lead to loss of utility. Therefore we investigate the stronger property of ex post individual rationality. Gresik [1991a] shows that for a general set of beliefs there exists an ex post individually rational mechanism that generates the same level of ex ante expected gains from trade as the interim individual rational mechanism that maximizes the ex ante expected gains from trade. We have the more ambitious goal to characterize all trading rules that are ex post individually rational-implementable. This goal is achieved for the special class of corner mechanisms when considering discrete uniform beliefs. This class consists of fairly simple mechanisms and despite this simple structure, it is still quite difficult to provide the full characterization. In general, we derive several necessary conditions for a mechanism that satisfy incentive compatibility and ex post individual rationality.

We also use our discrete bilateral trading model as a tool to do a robustness check for the continuous model. We compare the continuous model of Myerson and Satterthwaite [1983] with our discrete version to detect significant differences. There exists a modest discretisation effect when considering ex post efficiency. For some beliefs it is actually possible to achieve efficiency when considering a small number of different valuations, contrary to the result in the continuous case. Matsuo [1989] presents a necessary and sufficient condition for ex post efficient mechanisms to be feasible in the setting where both players have two types. Gresik [1991b] also analyzes the situation of two types. However in his paper it is allowed to have beliefs that are statistically dependent. He derives a necessary and sufficient condition for ex post efficient trade to be feasible. Our goal is to investigate what happens to efficiency if we have any finite number of independent types for each player. We show that in this setting ex post efficiency is usually not attainable.

The organization of the paper is as follows. Section 2 introduces the model. Incentive compatibility and individual rationality are introduced in Section 3 and 4, respectively. The main derivations and results can be found in Section 5 and 6. Section 7 discusses, among others, ex post efficiency. Most of the proofs can be found in the Appendix.

2 The model

Consider the following discrete bilateral trade model in which two players bargain under incomplete information. One player is the seller currently owning an indivisible object and the other player is the buyer, who is potentially willing to buy the object for a certain amount of money. Each player has $m + 1$ different possible valuations, namely: $0, 1, 2, \dots, m$ and for each player one of these valuations is drawn independently with a probability mass function $f_i(\cdot) > 0$ for $i = s, b$. For each $f_i(\cdot)$, we have a corresponding cumulative distribution function $F_i(\cdot)$.

Remark. Note that we could normalize our valuations such that all valuations are in between 0 and 1 (by division by m). By doing this, it becomes clear that an increase in m corresponds to a more precise approximation of the continuous case. Indeed, if we take the limit of m to infinity we obtain the continuous model.

Let v_s represent the valuation of the seller for the object and v_b represent the valuation of the buyer. Both are private information of the player. These two players are then going to participate in a bargaining mechanism. This mechanism determines whether there is going to be trade and for which price. We assume that each player learns his own valuation and that the prior probabilities of the player's valuations are common knowledge.

A direct bargaining mechanism is a mechanism in which each player simultaneously reports a valuation to a coordinator. The coordinator will then determine whether the object is traded and what the transfer will be. So each mechanism consists of three outcome matrices q , t^+ and t^- of size $(m+1) \times (m+1)$: here $q_{r_s, r_b} \in [0, 1]$ is the probability that the object is traded, $t_{r_s, r_b}^+ \in \mathbb{R}$ is the transfer of the buyer to the seller if there is trade and $t_{r_s, r_b}^- \in \mathbb{R}$ is the transfer of the buyer to the seller without trade, given that r_s and r_b respectively are the reported valuations. We refer to q as the trading rule and to (t^+, t^-) as the transfer scheme. Define $t_{r_s, r_b} = q_{r_s, r_b} \cdot t_{r_s, r_b}^+ + (1 - q_{r_s, r_b}) \cdot t_{r_s, r_b}^-$ for every r_s and r_b , to represent the expected transfer given the reported valuations r_s and r_b . Most of our results specify conditions for t . In those cases the actual choice of t^+ and t^- is not relevant, any choice of t^+ and t^- resulting in t is then feasible.

A direct mechanism is deterministic if $q_{r_s, r_b} \in \{0, 1\}$ for every r_s and r_b . In these mechanisms there is no need to make a distinction between the two different transfer matrices and thus we can solely concentrate on t . These are the type of mechanisms considered in the papers of Myerson and Satterthwaite [1983] and Gresik [1991a]. We, on the other hand, allow for uncertainty after the players reported their valuations. So in our analysis it is allowed to have a probabilistic mechanism in which $0 < q_{r_s, r_b} < 1$ for some r_s and r_b . For these mechanisms the distinction of the transfer matrices is useful in order to compare different notions of individual rationality. Financial derivatives are typical examples of commodities that exhibit this type of additional uncertainty.

In order for each player to be willing to participate in the bargaining mechanism, we need the mechanism to satisfy individual rationality. However, there exist several standard notions of individual rationality, namely ex ante, interim and ex post. Ex ante individual rationality requires that players commit to the trading mechanism before they learn their type, but do know the rules of the mechanism and the distribution of types of both players. Interim individual rationality is stronger and allows a player to drop out after learning his own type, but before he reports to the coordinator. Ex post individual rationality requires that even after both players reported their valuation, every player is still willing to participate. The strongest version is strong ex post individual rationality, which requires that money only transfers if the object is actually being traded with an amount in between both players' valuations.

A direct mechanism is (Bayesian) incentive compatible if honest reporting forms a Bayesian Nash equilibrium. This means that, in equilibrium, no type of player has an incentive to misreport to another type and thus it is optimal to report truthfully, given that the other player is expected to report truthfully. We analyze Bayesian incentive compatibility, as Hagerty and Rogerson [1987] showed that any mechanism satisfying dominant strategy incentive compatibility and ex post individual rationality is a posted-price mechanism. But this kind of mechanisms does not contain much flexibility and is too restrictive.

Without any loss of generality, it is possible to concentrate on direct mechanisms due to the revelation principle. This principle says that for any Bayesian equilibrium in any bargaining game, there exists an equivalent direct mechanism yielding the same outcome (both an equivalent allocation and payment) in which truthful reporting is a dominant strategy. Accordingly, we can reconstruct any equilibrium by the use of a direct mechanism.

In order to define the players' expected utilities for a given direct mechanism (q, t^+, t^-) , first

let

$$\begin{aligned}\bar{t}_s(r_s) &= \sum_{v_b=0}^m t_{r_s, v_b} \cdot f_b(v_b) & \bar{q}_s(r_s) &= \sum_{v_b=0}^m q_{r_s, v_b} \cdot f_b(v_b) \\ \bar{t}_b(r_b) &= \sum_{v_s=0}^m t_{v_s, r_b} \cdot f_s(v_s) & \bar{q}_b(r_b) &= \sum_{v_s=0}^m q_{v_s, r_b} \cdot f_s(v_s)\end{aligned}$$

All these quantities implicitly assume that the opponent reports truthfully. Here $\bar{t}_i(r_i)$ is the expected transfer and $\bar{q}_i(r_i)$ is the expected probability of trade for player $i = s, b$, given that r_i is the reported valuation. Note that since we consider expected quantities here, it is allowed to solely focus on t .

Then, given that the opponent reports truthfully, the expected utility of reporting r_i when having valuation v_i (for $i = s, b$) is

$$U_s(r_s, v_s) = \bar{t}_s(r_s) - v_s \cdot \bar{q}_s(r_s) \quad \text{and} \quad U_b(r_b, v_b) = v_b \cdot \bar{q}_b(r_b) - \bar{t}_b(r_b).$$

To simplify notation, let the expected utility when reporting truthfully be denoted by $U_i(v_i) = U_i(v_i, v_i)$.

3 Incentive compatibility

This section introduces the basic terminology and discusses preliminary results for incentive compatible mechanisms. We start with the definition of incentive compatibility.

Definition 3.1. A direct mechanism (q, t^+, t^-) is incentive compatible (IC) if

$$\text{for all } v_s \text{ and } r_s \quad U_s(v_s) \geq U_s(r_s, v_s) \quad \text{and}$$

$$\text{for all } v_b \text{ and } r_b \quad U_b(v_b) \geq U_b(r_b, v_b).$$

We call q implementable if there exists some (q, t^+, t^-) satisfying IC.

These inequalities assure that the expected utility does not increase when misreporting to different valuations r_s or r_b respectively. In other words, these are the conditions under which truthful reporting forms a Bayesian Nash equilibrium.

Definition 3.2. A direct mechanism (q, t^+, t^-) is weakly incentive compatible (WIC) if for $i = s, b$

$$\begin{aligned}U_i(0) &\geq U_i(1, 0) \\ U_i(m) &\geq U_i(m-1, m) \\ U_i(v_i) &\geq U_i(v_i-1, v_i) \quad \text{for all } v_i = 1, \dots, m-1 \\ U_i(v_i) &\geq U_i(v_i+1, v_i) \quad \text{for all } v_i = 1, \dots, m-1.\end{aligned}$$

So if a mechanism satisfies WIC no player has an incentive to misreport to those valuations adjacent to his own valuation. Clearly, any direct mechanism satisfying IC will also satisfy WIC. The following lemma shows that the converse is also true.

Lemma 3.3. *A direct mechanism (q, t^+, t^-) satisfies IC if and only if it satisfies WIC.*

The above result is useful since WIC is a property that is easier to check than IC, as we do not need to compare utilities among all different possible reports.

Now we present necessary conditions for any IC mechanism. Analogous to the continuous case of Myerson and Satterthwaite [1983], we show that the expected probability of trade, the expected utility and the expected payment are all weakly monotone.

Lemma 3.4. *Let the direct mechanism (q, t^+, t^-) be IC. Then*

(a) *the functions $\bar{q}_s(\cdot)$, $U_s(\cdot)$ and $\bar{t}_s(\cdot)$ are all weakly decreasing*

(b) *the functions $\bar{q}_b(\cdot)$, $U_b(\cdot)$ and $\bar{t}_b(\cdot)$ are all weakly increasing.*

Proof. Let (q, t^+, t^-) satisfy IC. By setting $r_s = v_s + 1 \leq m$ in Lemma A.1 in the Appendix, we are able to conclude from (9) that $\bar{q}_s(v_s + 1) \leq \bar{q}_s(v_s)$ and thus we find that $\bar{q}_s(\cdot)$ is weakly decreasing. Furthermore, since $\bar{q}_s(\cdot)$ is nonnegative we can also conclude that $U_s(v_s + 1) \leq U_s(v_s)$ and hence that $U_s(\cdot)$ is weakly decreasing.

In order to prove that $\bar{t}_s(\cdot)$ is weakly decreasing, observe that IC implies that for $v_s < m$

$$\begin{aligned} U_s(v_s) - U_s(v_s + 1, v_s) &= \bar{t}_s(v_s) - v_s \cdot \bar{q}_s(v_s) - \left(\bar{t}_s(v_s + 1) - v_s \cdot \bar{q}_s(v_s + 1) \right) \\ &= \bar{t}_s(v_s) - \bar{t}_s(v_s + 1) + v_s \cdot \left(\bar{q}_s(v_s + 1) - \bar{q}_s(v_s) \right) \\ &\geq 0. \end{aligned}$$

Since $\bar{q}_s(v_s + 1) \leq \bar{q}_s(v_s)$, the above inequality implies that $\bar{t}_s(v_s) - \bar{t}_s(v_s + 1) \geq 0$ and thus that $\bar{t}_s(\cdot)$ is weakly decreasing. The proofs for the buyer are analogous. \square

The above results are intuitive: the higher the valuation for the object of the seller, the less he is willing to trade. This implies a lower expected probability of trade and consequently also a lower revenue. Combined these last two statements result in a lower expected utility, as the effect of the decrease in expected probability of trade dominates. Likewise for the buyer: the lower his valuation, the less willing he is to acquire the object.

Given a direct mechanism (q, t^+, t^-) , we define the following quantities:

$$\begin{aligned} l_s &= \sum_{v_s=0}^{m-1} \sum_{v_i=v_s}^{m-1} \left(v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) + m \cdot \bar{q}_s(m) \right) \cdot f_s(v_s) + m \cdot \bar{q}_s(m) \cdot f_s(m) \\ u_s &= \sum_{v_s=0}^{m-1} \sum_{v_i=v_s}^{m-1} \left((v_i + 1) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) + m \cdot \bar{q}_s(m) \right) \cdot f_s(v_s) + m \cdot \bar{q}_s(m) \cdot f_s(m) \\ l_b &= \sum_{v_b=1}^m \sum_{v_j=1}^{v_b} (v_j - 1) \cdot \left(\bar{q}_b(v_j) - \bar{q}_b(v_j - 1) \right) \cdot f_b(v_b) \\ u_b &= \sum_{v_b=1}^m \sum_{v_j=1}^{v_b} v_j \cdot \left(\bar{q}_b(v_j) - \bar{q}_b(v_j - 1) \right) \cdot f_b(v_b). \end{aligned}$$

Notice that if q is implementable, we know that $l_s \leq u_s$ and $l_b \leq u_b$ due to the weak monotonicity property on $\bar{q}_s(\cdot)$ and $\bar{q}_b(\cdot)$ shown in Lemma 3.4. This implies that any implementable trading rule

q has two associated intervals, namely $[l_s, u_s]$ and $[l_b, u_b]$. These intervals will be used later on in Subsection 5.2.

For the upcoming statement, assume that the cumulative distribution of the seller $F_s(\cdot)$ is considered on the domain $\{-1, 0, 1, \dots, m\}$. We include -1 in the domain for ease of exposition, although this valuation always has a probability of 0.

Lemma 3.5.

$$u_b - l_s = \sum_{v_s=0}^m \sum_{v_b=0}^m \left(v_b - \frac{1 - F_b(v_b)}{f_b(v_b)} - v_s - \frac{F_s(v_s - 1)}{f_s(v_s)} \right) \cdot q_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b). \quad (1)$$

Lemma 3.5 explains the relation between $u_b - l_s$ and quantity (1) of Myerson and Satterthwaite [1983] or quantity $W(q)$ of Gresik [1991a]. This expression is important in the characterization of the set of interim individually rational-implementable trading rules. Bulow and Roberts [1989] already gave simple economic interpretations to the expressions

$$v_b - \frac{1 - F_b(v_b)}{f_b(v_b)} \quad \text{and} \quad v_s + \frac{F_s(v_s - 1)}{f_s(v_s)}.$$

They show that the former can be interpreted as the marginal revenue of the buyer, and the latter as the marginal cost of the seller. So $u_b - l_s$ can be seen as the ex ante expected difference between the marginal revenue and marginal cost of the two players.

Theorem 5.3 provides an interpretation of each of the quantities l_s , u_s , l_b and u_b . Namely, under the assumption that (q, t^+, t^-) is incentive compatibility and ex post individual rationality, these quantities form lower and respectively upper bounds for the expected transfer in the mechanism. So in fact, Lemma 3.5 gives a new interpretation to the above quantity: the difference between the highest possible expected transfer in the mechanism from the buyer's viewpoint and the lowest possible expected transfer in the mechanism from the seller's viewpoint, if the mechanism is incentive compatible and ex post individually rational.

The following lemma is comparable to the first part of Theorem 1 of Myerson and Satterthwaite [1983] and gives a lower and an upper bound for the sum of minimum expected utilities of both players in any IC mechanism. As we know that $U_s(\cdot)$ is weakly decreasing and $U_b(\cdot)$ is weakly increasing for every IC mechanism by Lemma 3.4, we can deduce that $U_s(m) = \min_{v_s} U_s(v_s)$ and $U_b(0) = \min_{v_b} U_b(v_b)$.

Lemma 3.6. *Let the direct mechanism (q, t^+, t^-) satisfy IC. Then*

$$l_b - u_s \leq U_s(m) + U_b(0) \leq u_b - l_s. \quad (2)$$

4 Individual rationality

In this section we define several types of individual rationality in order of appearance on the time line of the mechanism. Each type of individual rationality further down the time line of the mechanism implies all earlier types of individual rationality. This order can also be deduced from the amount of information each player acquires, because as the mechanism proceeds players obtain

more information. A natural definition to start with is ex ante individual rationality, which requires that each players' expected utility in the mechanism, averaged over all possible valuations, must be at least his expected utility without participation, which we assume to be zero. All of these definitions assume that players report truthfully.

Definition 4.1. A direct mechanism (q, t^+, t^-) is ex ante individually rational (EAIR) if

$$\sum_{v_s=0}^m U_s(v_s) \cdot f_s(v_s) \geq 0 \quad \text{and} \quad \sum_{v_b=0}^m U_b(v_b) \cdot f_b(v_b) \geq 0.$$

This concept only makes sense for mechanisms in which a player must choose to participate before he knows his own valuation for the object. However, we are more interested in stronger versions of individual rationality as we assume that each player learns his own valuation. Let us now state these definitions more formally.

Definition 4.2. A direct mechanism (q, t^+, t^-) is interim individually rational (IIR) if

$$\begin{aligned} U_s(v_s) &\geq 0 \quad \text{for all } v_s, \text{ and} \\ U_b(v_b) &\geq 0 \quad \text{for all } v_b. \end{aligned}$$

We call q IIR-feasible if there exists a transfer scheme (t^+, t^-) such that the resulting mechanism (q, t^+, t^-) satisfies IIR.

Observe that IIR implies EAIR. These inequalities assure that each player has a nonnegative utility from participating in the mechanism after learning his valuation.

Definition 4.3. A direct mechanism (q, t^+, t^-) is ex post individually rational (EPIR) if for all possible valuations v_s and v_b ,

$$t_{v_s, v_b} - v_s \cdot q_{v_s, v_b} \geq 0 \quad \text{and} \quad v_b \cdot q_{v_s, v_b} - t_{v_s, v_b} \geq 0.$$

We call q EPIR-feasible if there exists a transfer scheme (t^+, t^-) such that the resulting mechanism (q, t^+, t^-) satisfies EPIR.

Notice that EPIR implies IIR. These inequalities assure that if both players' reports are public information, each player's expected payoff is nonnegative. A direct conclusion we can draw from this is that the transfer will be in between both players' valuations if there is trade with certainty, i.e. $v_s \leq t_{v_s, v_b}^+ \leq v_b$ if $q_{v_s, v_b} = 1$, and that there will be no transfer if the probability of trade is zero, i.e. $t_{v_s, v_b}^- = 0$ if $q_{v_s, v_b} = 0$.

In all of the previous definitions of individual rationality it was possible to only concentrate on t since t can be seen as an expected payment after the players revealed their valuation, but before it is actually known whether there is going to be trade or not. In the strongest version of individual rationality, each player's expected utility needs to be nonnegative no matter the actual realisation of q , and as a result we need to impose restrictions on t^+ and t^- separately.

Definition 4.4. A mechanism (q, t^+, t^-) is strong ex post individually rational (SEPIR) if $v_s \leq t_{v_s, v_b}^+ \leq v_b$ whenever $q_{v_s, v_b} > 0$ and $t_{v_s, v_b}^- = 0$ whenever $q_{v_s, v_b} < 1$.

We call q SEPIR-feasible if there exists a transfer scheme (t^+, t^-) such that the resulting mechanism (q, t^+, t^-) satisfies SEPIR.

As a first observation notice that SEPIR implies EPIR. Secondly, as long as q is deterministic, ex post individual rationality and strong ex post individual rationality are equivalent. However, if $0 < q_{v_s, v_b} < 1$ for some valuations v_s and v_b , the difference becomes apparent. In this situation a lottery takes place after which trade occurs with a probability of q_{v_s, v_b} . The inequalities of an EPIR mechanism do not rule out the situation in which $t_{v_s, v_b}^- \neq 0$. In such a situation despite the negative outcome of the lottery, there will be a transfer of money without any trade of the object. If this is a payment of the buyer to the seller, the buyer has a negative utility from the trade. However, the precise conditions of this gamble were clear to both of them in advance, and the expected utility of this lottery for both players is nonnegative. For a SEPIR mechanism however, money transfers without trade are not allowed to happen.

Luckily, it is always possible to create a SEPIR mechanism if the inequalities in Definition 4.3 of an EPIR mechanism are satisfied.

Proposition 4.5. *If q is EPIR-feasible, then q is SEPIR-feasible.*

Proof. Suppose that (q, t^+, t^-) is EPIR. We will define transfer scheme $(\tilde{t}^+, \tilde{t}^-)$ in such a way that $\tilde{t}_{v_s, v_b} = t_{v_s, v_b}$ and $(q, \tilde{t}^+, \tilde{t}^-)$ satisfies SEPIR. If $q_{v_s, v_b} = 0$ and $q_{v_s, v_b} = 1$, let $\tilde{t}_{v_s, v_b} = t_{v_s, v_b}$. If $0 < q_{v_s, v_b} < 1$, then

$$\tilde{t}_{v_s, v_b}^+ = \frac{t_{v_s, v_b}}{q_{v_s, v_b}},$$

and $\tilde{t}_{v_s, v_b}^- = 0$. By the inequalities of Definition 4.3, $(q, \tilde{t}^+, \tilde{t}^-)$ is SEPIR. \square

Since our aim is to characterize implementable trading rules, we will only concentrate on the EPIR conditions in the rest of the paper.

5 Necessary conditions for IC and EPIR mechanisms

This section first provides necessary and sufficient conditions for a trading rule q to be interim individually rational-implementable. Thereafter, we derive necessary conditions for every incentive compatible and ex post individually rational mechanism.

5.1 Characterization of IIR-implementable trading rules

A trading rule q is IIR-implementable, if there exists a (q, t^+, t^-) satisfying both IC and IIR¹. The following theorem provides a characterization of IIR-implementable trading rules. Our characterization can be interpreted as a discrete version of Theorem 1 of Myerson and Satterthwaite [1983].

Theorem 5.1. *q is IIR-implementable if and only if $\bar{q}_s(\cdot)$ is weakly decreasing, $\bar{q}_b(\cdot)$ is weakly increasing and*

$$l_s \leq u_b. \tag{3}$$

¹There is a subtlety here. We already defined implementability of a trading rule. We also defined IIR-feasibility of a trading rule. However, the current definition forces us to use the same transfer rules t^+ and t^- in both definitions. In effect this is a stronger requirement than simple implementability together with IIR-feasibility.

5.2 Necessary conditions for IC and EPIR mechanisms

Now, we will derive necessary conditions for every EPIR-implementable trading rule. We will obtain our first result using the sum of minimum expected utilities of both players. Let us call q EPIR-implementable, if there exists a (q, t^+, t^-) satisfying both IC and EPIR.

Theorem 5.2. *Suppose q is EPIR-implementable. Then*

$$l_s \leq u_b \quad \text{and} \quad l_b \leq u_s. \quad (4)$$

Proof. Let (q, t^+, t^-) satisfy IC and EPIR. We know from Lemma 3.6 that

$$l_b - u_s \leq U_s(m) + U_b(0) \leq u_b - l_s,$$

whereas from Proposition A.2 in the appendix that

$$U_s(m) = \bar{t}_s(m) - m \cdot \bar{q}_s(m) = 0 \quad \text{and} \quad U_b(0) = -\bar{t}_b(0) = 0.$$

Combining them leads to (4). □

The following theorem presents a different necessary condition, namely one for the expected transfer in every IC and EPIR mechanism. This theorem is related to Lemma 3 of Gresik and Satterthwaite [1983], where payments for any IC mechanism are defined in the continuous model.

Theorem 5.3. *Let the direct mechanism (q, t^+, t^-) be IC and EPIR. Then*

$$l_s \leq \sum_{v_s=0}^m \sum_{v_b=0}^m t_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b) \leq u_s \quad \text{and} \quad l_b \leq \sum_{v_s=0}^m \sum_{v_b=0}^m t_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b) \leq u_b.$$

Proof. In this proof, we only show how to get

$$\sum_{v_s=0}^m \sum_{v_b=0}^m t_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b) \leq u_s.$$

All the other bounds can be derived using a similar method.

Let us construct the upper bound u_s by using the incentive compatibility constraints for different valuations of the seller. For each seller with a valuation of $v_s + 1$ for $v_s = 0, \dots, m-1$, IC implies

$$\begin{aligned} \bar{t}_s(v_s + 1) - (v_s + 1) \cdot \bar{q}_s(v_s + 1) &= U_s(v_s + 1) \\ &\geq U_s(v_s, v_s + 1) = \bar{t}_s(v_s) - (v_s + 1) \cdot \bar{q}_s(v_s), \end{aligned}$$

which can be rewritten to

$$\bar{t}_s(v_s) - \bar{t}_s(v_s + 1) \leq (v_s + 1) \cdot (\bar{q}_s(v_s) - \bar{q}_s(v_s + 1)).$$

By adding all these inequalities for $v_i = v_s, \dots, m-1$, we get

$$\bar{t}_s(v_s) - \bar{t}_s(m) \leq \sum_{v_i=v_s}^{m-1} (v_i + 1) \cdot (\bar{q}_s(v_i) - \bar{q}_s(v_i + 1)).$$

By property (d) of Proposition A.2 in the Appendix, we know that $\bar{t}_s(m) = m \cdot \bar{q}_s(m)$ and thus

$$\bar{t}_s(v_s) \leq \sum_{v_i=v_s}^{m-1} (v_i + 1) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) + m \cdot \bar{q}_s(m). \quad (5)$$

Remember, by definition

$$u_s = \sum_{v_s=0}^{m-1} \sum_{v_i=v_s}^{m-1} \left((v_i + 1) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) + m \cdot \bar{q}_s(m) \right) \cdot f_s(v_s) + m \cdot \bar{q}_s(m) \cdot f_s(m).$$

Adding the inequalities of (5) over all v_s and weighting by $f_s(v_s)$ yields

$$\sum_{v_s=0}^m \sum_{v_b=0}^m t_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b) = \sum_{v_s=0}^m \bar{t}_s(v_s) \cdot f_s(v_s) \leq u_s.$$

This completes the proof. \square

The above theorem shows that l_s , u_s , l_b and u_b are derived in a natural way by using the incentive compatibility constraints combined with one of the properties of every ex post individually rational mechanism.

Remember from Section 3 that any implementable trading rule has two associated intervals: $[l_s, u_s]$ and $[l_b, u_b]$. Notice that Theorem 5.3 implies that the intersection of the intervals $[l_s, u_s]$ and $[l_b, u_b]$ should be nonempty: $[l_s, u_s] \cap [l_b, u_b] \neq \emptyset$, in order for the expected transfer of an IC and EPIR mechanism to exist. Since any EPIR-feasible trading rule also satisfies IIR-feasibility, Theorem 5.1 implies that $l_s \leq u_b$. A consequence of these two observations is that $l_b \leq u_s$. Thus Theorem 5.2 can be proven in two different ways: via the expected utilities of the players (as in the proof of Theorem 5.2) or via the expected transfer in the mechanism (as in Theorem 5.3).

6 Deterministic and comprehensive trading rules

This section analyzes the class of deterministic and comprehensive trading rules. Let us first give the formal definition.

Definition 6.1. *A trading rule q is deterministic and comprehensive if for every v_s and v_b :*

- [1] $q_{v_s, v_b} \in \{0, 1\}$, and
- [2] $q_{v_s, v_b} = 1$ implies that $q_{v_i, v_j} = 1$ for all $v_i \leq v_s$ and all $v_j \geq v_b$.

Recall that in a deterministic mechanism it is known whether there is going to be trade after the reports are known to the coordinator. Remember that for these trading rules it is sufficient to analyze t instead of transfer scheme (t^+, t^-) .

A comprehensive mechanism is reasonable to consider, since it is a natural way to obtain the monotonicity properties of the expected probability of trade: a weakly decreasing $\bar{q}_s(\cdot)$ and a weakly increasing $\bar{q}_b(\cdot)$. As a result, we do not need to add monotonicity properties of $\bar{q}_i(\cdot)$ for $i = s, b$ as additional requirements in the following analysis.

6.1 Results related to IC+EPIR

The following theorem constructs a transfer t in such a way that money is only transferred in case of trade. This does not mean, yet, that these transfers are necessarily in between both players' valuations. Consequently, the resulting mechanism does not always satisfy EPIR.

Theorem 6.2. *Let q be a deterministic and comprehensive trading rule such that $l_s \leq u_b$ and $l_b \leq u_s$. Then, there exist a t such that (q, t^+, t^-) is IC and IIR, and moreover $U_s(m) = U_b(0) = 0$ and $t_{v_s, v_b} = 0$ if $q_{v_s, v_b} = 0$ for all v_s and v_b .*

Proof. Let us construct t such that (q, t^+, t^-) is IC and IIR with $U_s(m) = U_b(0) = 0$ and $t_{v_s, v_b} = 0$ if $q_{v_s, v_b} = 0$ for all v_s and v_b . By Theorem 5.1, q is IIR-implementable. Accordingly, we know that $l_s \leq u_s$ and $l_b \leq u_b$. Combined with the assumption that $l_s \leq u_b$ and $l_b \leq u_s$, it follows that there exists a T such that $l_s \leq T \leq u_s$ and $l_b \leq T \leq u_b$. Let us define $\Delta_s = \frac{T-l_s}{u_s-l_s}$ if $l_s < u_s$ and $\Delta_s = 0$ otherwise and $\Delta_b = \frac{T-l_b}{u_b-l_b}$ if $l_b < u_b$ and $\Delta_b = 0$ otherwise. Then we set $t_{v_s, v_b} = 0$ if $q_{v_s, v_b} = 0$ and

$$\text{[a] for } v_b = 1, \dots, m: \quad t_{m, v_b} = m \cdot q_{m, v_b}$$

$$\text{[b] for } v_s = 0, \dots, m: \quad t_{v_s, 0} = 0$$

$$\text{[c] for } v_s = 1, \dots, m-1 \text{ and } v_b = 1, \dots, m, \text{ if } q_{v_s, v_b} = 1:$$

$$t_{v_s, v_b} = \frac{\sum_{v_i=v_s}^{m-1} (v_i + \Delta_s) \cdot (\bar{q}_s(v_i) - \bar{q}_s(v_i + 1)) + m \cdot \bar{q}_s(m)}{\bar{q}_s(v_s)}$$

$$\text{[d] for } v_b = 1, \dots, m, \text{ if } q_{0, v_b} = 1:$$

$$t_{0, v_b} = \frac{\sum_{v_i=1}^{v_b} (v_i - 1 + \Delta_b) \cdot (\bar{q}_b(v_i) - \bar{q}_b(v_i - 1)) - \sum_{v_s=1}^m t_{v_s, v_b} \cdot f_s(v_s)}{f_s(0)}.$$

By construction, we know that $t_{v_s, v_b} = 0$ if $q_{v_s, v_b} = 0$ for all v_s and v_b . Cases [a] and [b] set prices in such a way that $U_s(m) = U_b(0) = 0$. Case [c] assures that prices are IC for the seller and case [d] makes sure that prices are IC for the buyer. The proof of these statements is given in the Appendix. \square

We proceed with an example of a deterministic and comprehensive trading rule q for which $l_s \leq u_b$ and $l_b \leq u_s$, while q is not EPIR-implementable. So even if we restrict our trading rule to the class of deterministic and comprehensive trading rules, the conditions $l_s \leq u_b$ and $l_b \leq u_s$ are not a sufficient condition for q to be an EPIR-implementable trading rule.

Example. Assume that valuations are discrete uniformly distributed, that is $f_i(v_i) = \frac{1}{m+1}$ for all valuations v_i for $i = s, b$. Consider Trading rule 1. Let us show that it is not possible for this mechanism to satisfy both IC and EPIR. Assume (q, t^+, t^-) is IC and EPIR. We derive a contradiction. By the type of seller with valuation 0,

$$\sum_{v_b=5}^8 t_{0, v_b} \leq 11.$$

	11	8	8	8	8	8	8	8	0		
	7	7	7	7	7	7	7	7	0		
8	1	1	1	1	1	1	1	1	0	53	61
7	1	0	0	0	0	0	0	0	0	4	5
6	1	0	0	0	0	0	0	0	0	4	5
5	1	0	0	0	0	0	0	0	0	4	5
4	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8		

Trading rule 1: The first column specifies the different valuations for each type of buyer, and the last row specifies the different valuations for each type of seller. The inner matrix forms the trading rule q . So each column in this matrix represents a type of seller and each row a type of buyer. The first two rows specify the upper and lower bounds for the sum of prices for that specific type of seller in the corresponding column. Similarly, the last two columns specify the upper and lower bounds for the sum of prices for that specific type of buyer in the row. How to obtain these bounds for the sum of prices for the seller? For the lower bound: sum the maximum value the seller could have reported while still having trade, over all type of buyers he trades with. Whereas for the upper bound we sum the minimum value that could be reported without having trade, over all type of buyers he has trade with (which is equivalent to inequality (5)). For example, a seller with a valuation of 1 only has trade with a buyer with valuation of 8. For this type of buyer, the maximum valuation that the seller could have reported while still having trade is 7 and the minimum reported valuation without having trade is 8.

By the types of buyers with valuations 5, 6 and 7,

$$\sum_{v_b=5}^7 t_{0,v_b} \geq 12.$$

Since $t_{0,8} \geq 0$, we obtained a contradiction.

6.2 Corner mechanisms

In this and the following subsection it is assumed that the valuations are discrete uniformly distributed, that is $f_i(v_i) = \frac{1}{m+1}$ for all valuations v_i for $i = s, b$. Let us first present a lemma that shows that we have a symmetric problem.

Lemma 6.3. ² *Let $f_i(v_i) = \frac{1}{m+1}$ for all valuations v_i for $i = s, b$ and q be deterministic and comprehensive. If q is EPIR-implementable, then q^T is also EPIR-implementable.*

This subsection considers a special class of deterministic and comprehensive mechanisms with a simple structure, the so-called corner mechanisms, and gives a full characterization of all EPIR-implementable trading rules contained in this class.

²We define $q_{v_s, v_b}^T = q_{m-v_b, m-v_s}$

Consider the following class of deterministic and comprehensive trading rules: q a corner mechanism if for $k, \ell \in \mathbb{N}$ and $k \leq m$ and $\ell \leq m$, we have

m	1	1	...	1	0	...	0
$m-1$	1	0	...	0	0	...	0
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots
$m-\ell$	1	0	...	0	0	...	0
$m-\ell-1$	0	0	...	0	0	...	0
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots	\ddots	\vdots
0	0	0	...	0	0	...	0
	0	1	...	k	$k+1$...	m

For instance, Trading rule 1 in Subsection 6.1 is a corner mechanism. Lemma A.3 in the Appendix shows that there at most three different prices present in such mechanism if we assume IC and EPIR. Moreover, the lemma presents necessary and sufficient conditions for these prices.

The following theorem presents necessary and sufficient conditions for a corner mechanism q to be EPIR-implementable. With the addition of two constraints to the two known constraints, $l_s \leq u_b$ and $l_b \leq u_s$ from Theorem 5.2, we obtain a full characterization.

Theorem 6.4. *Assume $f_i(v_i) = \frac{1}{m+1}$ for all valuations v_i for $i = s, b$ and let q be a corner matrix. Then q is EPIR-implementable if and only if $l_s \leq u_b$, $l_b \leq u_s$, and whenever $q_{1,m} = q_{0,m-1} = 1$:*

$$\sum_{v_s=0}^{m-1} q_{v_s,m} - \sum_{v_b=1}^{m-1} (2v_b - m - 1) \cdot q_{0,v_b} \geq 0 \quad (6)$$

and

$$\sum_{v_s=0}^{m-1} (m - 2v_s - 1) \cdot q_{v_s,m} - \sum_{v_b=1}^{m-1} q_{0,v_b} - q_{m,m} \leq m. \quad (7)$$

Proof. ³ First, observe that

$$\sum_{v_s=0}^{m-1} q_{v_s,m} - \sum_{v_b=1}^{m-1} (2v_b - m - 1) \cdot q_{0,v_b} = \ell + k + 1 - q_{m,m} - (m - \ell - 1 + q_{0,0}) \cdot \ell$$

and

$$\sum_{v_s=0}^{m-1} (m - 2v_s - 1) \cdot q_{v_s,m} - \sum_{v_b=1}^{m-1} q_{0,v_b} - q_{m,m} = m - \ell - 1 + q_{0,0} + k \cdot (m - 1) - k \cdot (k + 1 - q_{m,m}).$$

We prove the “only if” part. Assume q is EPIR-implementable. From Theorem 5.2 we know that $l_s \leq u_b$ and $l_b \leq u_s$. We can deduce (6) by using lemma A.3:

$$\begin{aligned} 0 &\leq z \\ &= z + \ell \cdot y - x + x - \ell \cdot y \\ &\leq \ell + k + 1 - q_{m,m} - (m - \ell - 1 + q_{0,0}) \cdot \ell, \end{aligned}$$

³The proof uses the computational results from Subsection 6.3. Although all of these cases can also be checked by hand.

where the first inequality follows from (16) and the second inequality from the combination of (13), (14) and (15). In a similar way, we can deduce (7):

$$\begin{aligned} m &\geq z \\ &= z + k \cdot x - y + y + k \cdot x \\ &\geq m - \ell - 1 + k \cdot (m - 1) - k \cdot (k + 1 - q_{m,m}), \end{aligned}$$

where the first inequality follows from (16) and the second inequality from the combination of (12), (14) and (15).

In order to complete the “if part” of the proof, suppose that $l_s \leq u_b$, $l_b \leq u_s$ and that inequalities (6) and (7) hold. Because of the symmetry of the problem shown in Lemma 6.3, we only consider all different possible values for k . The same analysis then also holds for l by using constraint (7).

The computational results of Subsection 6.3 show that for $m < 8$ all deterministic and comprehensive trading rules satisfying $l_s \leq u_b$ and $l_b \leq u_s$ are EPIR-implementable and thus w.l.o.g. we can assume that $m \geq 8$.

Assume $k = 0$, then setting $t_{0, \frac{m-\ell}{m}} = \dots = t_{0,1} = \frac{m-\ell-1+q_{0,0}}{m}$ results in a mechanism satisfying IC and EPIR.

Now assume $k = m$, then we know from $l_s \leq u_b$ and symmetry that $0 < \ell < m$. From (6), we can deduce that $m + \ell \geq (m - \ell - 1) \cdot \ell$. In order to use Lemma A.3 of the Appendix, we distinguish three cases:

1. if $(m - \ell - 1) \cdot \ell \leq m + \ell \leq (m - \ell - 1) \cdot (\ell + 1)$.
Set $x = m$, $y = m - \ell - 1$ and $z = m + \ell - (m - \ell - 1) \cdot \ell$
2. if $(m - \ell - 1) \cdot (\ell + 1) < m + \ell < (m - \ell) \cdot (\ell + 1)$.
Set $x = m$ and $y = z = \frac{m+\ell}{\ell+1}$
3. if $(m - \ell) \cdot (\ell + 1) \leq m + \ell$.
Set $x = m$ and $y = z = m - \ell$.

Notice for case 1 that $0 \leq z \leq m - \ell - 1$, for case 2 that $m - \ell - 1 < y = z < m - \ell$ and for case 3 that $u_b \leq u_s$ in order to check that for all of these cases x , y and z satisfy the inequalities of Lemma A.3 and thus generate a mechanism that satisfies IC and EPIR.

If $k = m - 1$, we only need to consider the cases $0 < \ell < m$ due to symmetry and hence we can distinguish the same three cases as above, with accompanying x , y and z .

For $0 < k \leq m - 2$, we know due to symmetry that we only need to consider $0 < \ell \leq m - 2$, but then $l_s \leq u_b$ and $l_b \leq u_s$ assure that only two cases remain: $k = m - 2$ and $\ell = m - 2$ or $k = m - 3$ and $\ell = m - 2$. Setting $x = z = k + 1$ and $y = m - \ell - 1$ works for both cases. \square

Hence the addition of two extra constraints is sufficient to obtain the full characterization the set of EPIR-implementable corner mechanisms. Furthermore, none of these constraints are redundant. We describe those cases for which (6) is not redundant. Or to state it in different words, we describe those cases for which all constraints are satisfied except (6). Because of symmetry, the same holds for (7) by reversing k and ℓ . Let $k = m$ or $k = m - 1$. Then in the following situations (6) is not redundant:

- (a) if $m = 8$ and $\ell = 3$

(b) if $m = 9$ and $2 \leq \ell \leq 5$

(c) if $m \geq 9$ and $\frac{m-2-\sqrt{m^2-8m+4}}{2} < \ell \leq \frac{m-3-\sqrt{m^2-10m+5}}{2}$ or $\frac{m-3+\sqrt{m^2-10m+5}}{2} \leq \ell < \frac{m-2+\sqrt{m^2-8m+4}}{2}$.

However for $m \geq 11$, there is only one integer value that actually lies within these bounds. Hence there are not that many trading rules for which $l_s \leq u_b$ and $l_b \leq u_s$, while not satisfying the two additional constraints. Most of the examples mentioned in Subsection 6.3 are of this kind.

6.3 Computational results

This subsection considers deterministic and comprehensive mechanisms with discrete uniformly distributed valuations. In Subsection 6.1, we presented an example of a deterministic and comprehensive trading rule for with $l_s \leq u_b$ and $l_b \leq u_s$ that is not implementable by an EPIR transfer scheme. The program package Mathematica was used to find more of such examples. Let us first show a short summary of the results obtained using Mathematica:

# valuations	# det+compr	#IIR	#EPIR	$l_s \leq u_b$ & $l_b \leq u_s$	#examples
1	2	2	2	2	0
2	5	4	4	4	0
3	14	10	10	10	0
4	42	24	23	23	0
5	132	67	63	63	0
6	429	184	169	169	0
7	1430	541	493	493	0
8	4862	1580	1423	1423	0
9	16796	4785	4316	4320	4
10	58786	14496	12901	12919	18
11	208012	45209	39997	40029	32

The first column shows the number of different possible valuations for each player (which equals $m + 1$ in our setting). The second column shows the amount of deterministic and comprehensive trading rules q for each number of valuations. The third and fourth column represent the number of implementable trading rules satisfying IIR-feasibility, and EPIR-feasibility respectively. The last column illustrates the number of deterministic and comprehensive trading rules with $l_s \leq u_b$ and $l_b \leq u_s$ that cannot be implemented by an EPIR transfer scheme.

As long as the number of possible valuations is less than or equal to eight, all deterministic and comprehensive trading rules satisfying $l_s \leq u_b$, $l_b \leq u_s$ are EPIR-implementable. So examples only exist if each player has more than eight different possible valuations. However compared to the total number of trading rules, the number of examples is relatively small.

Now that we have seen how to characterize trading rules for corner mechanisms, let us present a different kind of example. Special for this trading rule is that quite some of the prices are fixed due to the combination of IC and EPIR. These type of examples exist for $m \geq 9$.

Example. ⁴ Consider Trading rule 2. Let us show that it is not possible for this mechanism to satisfy IC and EPIR.

⁴All the examples we found using Mathematica are similar to the two examples discussed so far.

	21	18	18	18	18	18	18	18	18	9		
	17	17	17	17	17	17	17	17	17	9		
9	1	1	1	1	1	1	1	1	1	1	68	78
8	1	1	1	1	1	1	1	1	1	0	60	69
7	1	0	0	0	0	0	0	0	0	0	4	5
6	1	0	0	0	0	0	0	0	0	0	4	5
5	1	0	0	0	0	0	0	0	0	0	4	5
4	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9		

Trading rule 2: The trading rule should be interpreted similar as Trading rule 1.

Assume (q, t^+, t^-) is IC and EPIR. We derive a contradiction. For $v_s = 1, \dots, 8$, we know due to EPIR that $t_{v_s,8} \leq 8$ and $t_{v_s,9} \leq 9$ and from the lower bound that $t_{v_s,8} + t_{v_s,9} \geq 17$. This implies that $t_{v_s,8} = 8$ and $t_{v_s,9} = 9$ for $v_s = 1, \dots, 8$. Thus

$$\sum_{v_s=1}^9 t_{v_s,9} = 81.$$

However from the type of buyer with valuation 1, we know that

$$\sum_{v_s=0}^9 t_{v_s,9} \leq 78.$$

Since $t_{0,9} \geq 0$, we have a contradiction.

7 Other issues

7.1 Ex post efficiency

From a social point of view it is optimal to always have trade if the buyer values the object higher and to never have trade if the seller values the object higher. This is called ex post efficiency.

Definition 7.1. A trading rule q is ex post efficient (EPE) if $q_{v_s, v_b} = 1$ whenever $v_b > v_s$ and $q_{v_s, v_b} = 0$ whenever $v_b < v_s$.

Clearly, for all EPE trading rules $\bar{q}_s(\cdot)$ is weakly decreasing and $\bar{q}_b(\cdot)$ is weakly increasing. However, it is not specified what needs to be done in case of equal valuations. Therefore, we analyze the two extreme cases: always and respectively never having trade if both valuations are equal.

Theorem 7.2. Let q be EPE. If $q_{v_s, v_b} = 1$ for all valuations $v_s = v_b$, then q is not IIR-implementable.

Proof. Assume that q is EPE and that $q_{v_s, v_b} = 1$ if $v_b = v_s$. In order to find out whether this q is IIR-implementable, let us check inequality (3):

$$\begin{aligned}
& u_b - l_s \\
&= \sum_{v_s=0}^m \sum_{v_b=v_s}^m \left(v_b - \frac{1 - F_b(v_b)}{f_b(v_b)} - v_s - \frac{F_s(v_s - 1)}{f_s(v_s)} \right) \cdot f_s(v_s) \cdot f_b(v_b) \\
&= \sum_{v_s=0}^m \sum_{v_b=v_s}^m \left(v_b \cdot f_b(v_b) - 1 + F_b(v_b) \right) \cdot f_s(v_s) - \sum_{v_s=0}^m \sum_{v_b=v_s}^m \left(v_s \cdot f_s(v_s) + F_s(v_s - 1) \right) \cdot f_b(v_b) \\
&= \sum_{v_s=0}^m v_s \cdot f_s(v_s) \cdot \left(1 - F_b(v_s - 1) \right) - \sum_{v_s=0}^m \left(v_s \cdot f_s(v_s) + F_s(v_s - 1) \right) \cdot \left(1 - F_b(v_s - 1) \right) \\
&= - \sum_{v_s=0}^m F_s(v_s - 1) \cdot \left(1 - F_b(v_s - 1) \right).
\end{aligned}$$

Hence, in this case we will always get a negative outcome and therefore q is not IIR-implementable according to Theorem 5.1. \square

The above result is in line with the continuous case. The above quantity can be seen as the lump-sum subsidy necessary to assure that an EPE implementable trading rules is IIR-feasible.

However, if $q_{v_s, v_b} = 0$ for all valuations $v_s = v_b$, we obtain a slightly different result:

$$\begin{aligned}
& u_b - l_s \\
&= \sum_{v_s=0}^m \sum_{v_b=v_s+1}^m \left(v_b - \frac{1 - F_b(v_b)}{f_b(v_b)} - v_s - \frac{F_s(v_s - 1)}{f_s(v_s)} \right) \cdot f_s(v_s) \cdot f_b(v_b) \\
&= \sum_{v_s=0}^m \sum_{v_b=v_s+1}^m \left(v_b \cdot f_b(v_b) - 1 + F_b(v_b) \right) \cdot f_s(v_s) - \sum_{v_s=0}^m \sum_{v_b=v_s+1}^m \left(v_s \cdot f_s(v_s) + F_s(v_s - 1) \right) \cdot f_b(v_b) \\
&= \sum_{v_s=0}^m (v_s + 1) \cdot f_s(v_s) \cdot \left(1 - F_b(v_s) \right) - \sum_{v_s=0}^m \left(v_s \cdot f_s(v_s) + F_s(v_s - 1) \right) \cdot \left(1 - F_b(v_s) \right) \\
&= - \sum_{v_s=0}^m \left(-f_s(v_s) + F_s(v_s - 1) \right) \cdot \left(1 - F_b(v_s) \right).
\end{aligned}$$

This expression need not be negative, even though it often will be.

Example. Consider the following setting: $f_i(v_i) = \frac{1}{m+1}$ for all valuations v_i for $i = s, b$ and let $m = 4$. The above expression then equals 0 and it can be checked that (q, t^+, t^-) is IC and EPIR, where

$$t = \begin{array}{c|ccccc}
4 & 2 & 2 & 3 & 3 & 0 \\
3 & 2 & 2 & 2 & 0 & 0 \\
2 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\hline
& 0 & 1 & 2 & 3 & 4
\end{array}.$$

It can be checked that in the discrete uniform case for $m > 4$, any implementable trading rule is not IIR-feasible and therefore also not EPIR-feasible. So only for small values of m we are able to find EPE mechanisms that satisfy IC and EPIR.

7.2 Maximizing expected total gains from trade

The expected total gains from trade in a mechanism is

$$\sum_{v_s=0}^m U_s(v_s) \cdot f_s(v_s) + \sum_{v_b=0}^m U_b(v_b) \cdot f_b(v_b) = \sum_{v_s=0}^m \sum_{v_b=0}^m (v_b - v_s) \cdot q_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b).$$

Since ex post efficiency is, in most cases, unattainable (see Theorem 7.2), it is natural to seek for the mechanism that maximizes the total expected gains from trade subject to incentive compatibility and individual rationality. In order to find this mechanism, analogous to the terminology in Myerson and Satterthwaite [1983] and Gresik [1991a], we define for every $\alpha \in [0, 1]$:

$$q_{v_s, v_b}^\alpha = \begin{cases} 1 & \text{if } v_s + \alpha \frac{F_s(v_s-1)}{f_s(v_s)} \leq v_b - \alpha \frac{1-F_b(v_b)}{f_b(v_b)}, \\ 0 & \text{if } v_s + \alpha \frac{F_s(v_s-1)}{f_s(v_s)} > v_b - \alpha \frac{1-F_b(v_b)}{f_b(v_b)}. \end{cases}$$

The following theorem is analogous to Theorem 2 of Myerson and Satterthwaite [1983].

Theorem 7.3. *If there exists an IC mechanism (q, t^+, t^-) such that $U_s(m) = U_b(0) = 0$ and $q = q^\alpha$ for some $\alpha \in [0, 1]$, then this mechanism maximizes the expected total gains from trade among all IC and IIR mechanisms.*

8 Linear mechanisms

In this section we analyze a particular class of deterministic and comprehensive trading rules as illustrative example. Assume that valuations are discrete uniformly distributed, that is $f_i(v_i) = \frac{1}{m+1}$ for all valuations v_i for $i = s, b$. Consider the following class of trading rules q with $n \in \mathbb{N}$, where $n \leq m$,

m	1	1	...	1	1	0	...	0
$m-1$	1	1	...	1	0	0	...	0
\vdots	\vdots	\vdots	\ddots		\vdots	\vdots		\vdots
$m-n+2$	1	1	...	0	0	0	...	0
$m-n+1$	1	0	...	0	0	0	...	0
$m-n$	0	0	...	0	0	0	...	0
\vdots	\vdots	\vdots		\vdots	\vdots	\vdots		\vdots
0	0	0	...	0	0	0	...	0
	0	1	...	$n-2$	$n-1$	n	...	m

In these trading rules n represents the largest number of opponent's types with which a type has trade (so the maximum amount of one's in each row and column):

$$n = \max_{v_s} (m+1) \cdot \bar{q}_s(v_s) = \max_{v_b} (m+1) \cdot \bar{q}_b(v_b).$$

Then by the formula given in the previous section, we get the following values for the upper and lower bounds of the expected transfer:

$$\begin{aligned} l_s &= \frac{(n+1) \cdot n \cdot (n-1)}{3 \cdot (m+1)^2} \\ u_s &= \frac{(2n+1) \cdot (n+1) \cdot n}{6 \cdot (m+1)^2} \\ l_b &= \frac{(3m-2n-1) \cdot (n+1) \cdot n}{6 \cdot (m+1)^2} \\ u_b &= \frac{(3m-2n+2) \cdot (n+1) \cdot n}{6 \cdot (m+1)^2}. \end{aligned}$$

What should be noticed is that the length of both of the intervals between the upper and lower bound (scaled in the proper way) is equal to $(m+1)^2 \cdot (u_s - l_s) = (m+1)^2 \cdot (u_b - l_b) = \frac{n \cdot (n+1)}{2}$. This difference is equal to the total number of combinations of types for which trade occurs (total number of one's in our trading rule). The main question we want to answer is: for a given $m \in \mathbb{N}$, for which n is q EPIR-implementable? Distinguish the following three cases:

- (i) If $n > \frac{3}{4}m + 1$, we get that $l_b < u_b < l_s < u_s$. From Theorem 5.1 we can conclude that q is not IIR-implementable.
- (ii) If $n < \frac{3}{4}m - \frac{1}{2}$, we have that $l_s < u_s < l_b < u_b$ and hence according to Theorem 5.3 q is not EPIR-implementable. However as $u_b > l_s$, according to theorem 5.1 it is possible to find a (q, t^+, t^-) satisfying IC and IIR.
- (iii) If $\frac{3}{4}m - \frac{1}{2} \leq n \leq \frac{3}{4}m + 1$, there may exists a (q, t^+, t^-) satisfying IC and EPIR. In fact, such mechanism exists. Namely, the mechanism (q, t^+, t^-) , where

$$t_{v_s, v_b} = \begin{cases} (v_s + v_b + \frac{m}{2}) / 3 & \text{if } q_{v_s, v_b} = 1, \\ 0 & \text{else,} \end{cases}$$

is both IC and EPIR.

8.1 Maximizing expected total gains from trade

From Subsection 7.2, we know which trading rule maximizes the total expected gains from trade:

$$q_{v_s, v_b}^\alpha = \begin{cases} 1 & \text{if } v_s \leq v_b - \frac{m \cdot \alpha}{1 + \alpha}, \\ 0 & \text{if } v_s > v_b - \frac{m \cdot \alpha}{1 + \alpha}. \end{cases}$$

In order to find α , we need to use that $U_s(m) = U_b(0) = 0$. Using Theorem 5.1, we get

$$\begin{aligned} 0 &\leq u_b - l_s \\ &= \sum_{v_s=0}^m \sum_{v_b=0}^m \frac{2v_b - 2v_s - m}{(m+1)^2} \cdot q_{v_s, v_b}^\alpha \\ &\leq \frac{m \cdot (-1 + 3\alpha) \cdot (1 + \alpha + m) \cdot (2 + 2\alpha + m)}{6 \cdot (1 + \alpha)^3 \cdot (m+1)^2}. \end{aligned}$$

Equality is assured by setting $\alpha = \frac{1}{3}$. The trading rule solving this problem is

$$q_{v_s, v_b}^{\frac{1}{3}} = \begin{cases} 1 & \text{if } v_s \leq v_b - \frac{m}{4}, \\ 0 & \text{if } v_s > v_b - \frac{m}{4}, \end{cases}$$

which is equivalent to the linear mechanism with $n = \frac{3m+4}{4}$. If we standardize this trading rule such that all valuations are in between 0 and 1, we found the discrete version of the mechanism found by Chatterjee and Samuelson [1983]. These results are in line with the results from Gresik [1991a], saying that for a general set of beliefs there exists a transfer t such that the ex ante expected gains from trade maximizing mechanism is IC and EPIR.

9 Conclusion

We provide an elaborate study of the bilateral trade model when having discrete valuations. Our most important goal was to try to find a characterization of every trading rule that is EPIR-feasible. We focus on ex post individual rationality as the majority of research on this topic does not incorporate this stronger notion of individual rationality. Although potential buyers could be hesitant to participate in a mechanism that can leave him worse off compared to what they begin with. By using this stronger notion of individual rationality, we obtained the link between the sum of minimum expected utilities and the expected transfer in a mechanism. An interesting question for future research is to see what happens if we abandon the assumption of discrete uniformly distributed valuations in order to find a more general characterization.

Another interesting topic is to analyze the setting in which the valuations are not independently distributed. Although there is already some literature on this topic, it is still interesting to find out whether efficiency can be restored and more importantly how.

A Appendix

In this Appendix, the reader will find proofs not given in the text and the remaining lemmas and propositions.

Proof of Lemma 3.3. The “only if” part is clear from the definition of IC.

For the “if” part it is shown by an inductive argument that if we assume WIC, a seller with a valuation of v_s does not want to misreport to a higher valuation. A similar argument shows that no type of seller wants to deviate to a lower valuation, meaning that the mechanism is incentive compatible for the seller. In an analogous way, it is possible to prove that the mechanism will also be incentive compatible for the buyer.

We will show by induction that $U_s(v_s) \geq U_s(r_s, v_s)$ for every report $r_s > v_s$. Pick an arbitrary valuation v_s for $v_s = 0, \dots, m-1$ and some arbitrary report $r_s > v_s$ and assume WIC. For the base case notice that if $r_s = v_s + 1$, WIC directly implies that $U_s(v_s) \geq U_s(v_s + 1, v_s)$.

For the induction step, we need to prove that if $U_s(v_s) \geq U_s(r_s, v_s)$ for $v_s < r_s < m$, then it

must also hold that $U_s(v_s) \geq U_s(r_s + 1, v_s)$. Observe that the following holds due to WIC:

$$\begin{aligned}
U_s(r_s) &\geq U_s(r_s + 1, r_s) \\
&= \bar{t}_s(r_s + 1) - r_s \cdot \bar{q}_s(r_s + 1) \\
&= U_s(r_s + 1) + \bar{q}_s(r_s + 1) \\
&\geq U_s(r_s, r_s + 1) + \bar{q}_s(r_s + 1) \\
&= \bar{t}_s(r_s) - (r_s + 1) \cdot \bar{q}_s(r_s) + \bar{q}_s(r_s + 1) \\
&= U_s(r_s) - \bar{q}_s(r_s) + \bar{q}_s(r_s + 1)
\end{aligned}$$

and thus we can conclude that

$$\bar{q}_s(r_s) \geq \bar{q}_s(r_s + 1). \quad (8)$$

So if we assume that $U_s(v_s) \geq U_s(r_s, v_s)$, we have

$$\begin{aligned}
U_s(v_s) &\geq U_s(r_s, v_s) \\
&= \bar{t}_s(r_s) - v_s \cdot \bar{q}_s(r_s) \\
&= U_s(r_s) + (r_s - v_s) \cdot \bar{q}_s(r_s) \\
&\geq U_s(r_s + 1, r_s) + (r_s - v_s) \cdot \bar{q}_s(r_s) \\
&= \bar{t}_s(r_s + 1) - r_s \cdot \bar{q}_s(r_s + 1) + (r_s - v_s) \cdot \bar{q}_s(r_s) \\
&\geq \bar{t}_s(r_s + 1) - r_s \cdot \bar{q}_s(r_s + 1) + (r_s - v_s) \cdot \bar{q}_s(r_s + 1) \\
&= \bar{t}_s(r_s + 1) - v_s \cdot \bar{q}_s(r_s + 1) \\
&= U_s(r_s + 1, v_s),
\end{aligned}$$

where the third inequality follows from inequality (8) and the fact that $v_s < r_s$. \square

The following lemma describes the expected utility of each type of player in a IC mechanism.

Lemma A.1. *If the direct mechanism (q, t^+, t^-) is IC. Then for all valuations $v_s < r_s$ and $v_b > r_b$*

$$U_s(r_s) + \sum_{v_i=v_s+1}^{r_s} \bar{q}_s(v_i) \leq U_s(v_s) \leq U_s(r_s) + \sum_{v_i=v_s}^{r_s-1} \bar{q}_s(v_i) \quad (9)$$

$$U_b(r_b) + \sum_{v_j=r_b}^{v_b-1} \bar{q}_b(v_j) \leq U_b(v_b) \leq U_b(r_b) + \sum_{v_j=r_b+1}^{v_b} \bar{q}_b(v_j). \quad (10)$$

Proof. Let us prove the left inequality of (9). The rest can be proven analogously.

We prove by induction that for $v_s < r_s \leq m$:

$$U_s(v_s) - U_s(r_s) \geq \sum_{v_i=v_s+1}^{r_s} \bar{q}_s(v_i).$$

For the base case $r_s = v_s + 1$, we get from WIC that

$$\begin{aligned}
U_s(v_s) - U_s(v_s + 1) &\geq U_s(v_s + 1, v_s) - U_s(v_s + 1) \\
&= \bar{t}_s(v_s + 1) - v_s \cdot \bar{q}_s(v_s + 1) - \left(\bar{t}_s(v_s + 1) - (v_s + 1) \cdot \bar{q}_s(v_s + 1) \right) \\
&= \bar{q}_s(v_s + 1).
\end{aligned}$$

Now assume that the induction step holds for $r_s < m$, then

$$\begin{aligned}
U_s(v_s) - U_s(r_s + 1) &= U_s(v_s) - U_s(r_s) + U_s(r_s) - U_s(r_s + 1) \\
&\geq \sum_{v_i=v_s+1}^{r_s} \bar{q}_s(v_i) + \bar{q}_s(r_s + 1) \\
&= \sum_{v_i=v_s+1}^{r_s+1} \bar{q}_s(v_i),
\end{aligned}$$

which proves the inequality for $r_s + 1$. □

Proof of Lemma 3.5. Let us prove equality (1):

$$\begin{aligned}
&u_b - l_s \\
&= \sum_{v_b=1}^m \sum_{v_j=1}^{v_b} v_j \cdot \left(\bar{q}_b(v_j) - \bar{q}_b(v_j - 1) \right) \cdot f_b(v_b) \\
&\quad - \sum_{v_s=0}^{m-1} \sum_{v_i=v_s}^{m-1} \left(v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) + m \cdot \bar{q}_s(m) \right) \cdot f_s(v_s) - m \cdot \bar{q}_s(m) \cdot f_s(m) \\
&= \sum_{v_b=1}^m \left(- \sum_{v_j=0}^{v_b-1} \bar{q}_b(v_j) + v_b \cdot \bar{q}_b(v_b) \right) \cdot f_b(v_b) - \sum_{v_s=0}^{m-1} \left(v_s \cdot \bar{q}_s(v_s) + \sum_{v_i=v_s+1}^m \bar{q}_s(v_i) \right) \cdot f_s(v_s) - m \cdot \bar{q}_s(m) \cdot f_s(m) \\
&= \sum_{v_b=0}^m \left(v_b - \frac{1 - F_b(v_b)}{f_b(v_b)} \right) \cdot \bar{q}_b(v_b) \cdot f_b(v_b) - \sum_{v_s=0}^m \left(v_s + \frac{F_s(v_s - 1)}{f_s(v_s)} \right) \cdot \bar{q}_s(v_s) \cdot f_s(v_s) \\
&= \sum_{v_s=0}^m \sum_{v_b=0}^m \left(v_b - \frac{1 - F_b(v_b)}{f_b(v_b)} - v_s - \frac{F_s(v_s - 1)}{f_s(v_s)} \right) \cdot q_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b).
\end{aligned}$$

□

Proof of Lemma 3.6. In order to find that $U_s(m) + U_b(0) \leq u_b - l_s$, we rewrite the following

expression:

$$\begin{aligned}
& \sum_{v_s=0}^m \sum_{v_b=0}^m (v_b - v_s) \cdot q_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b) \\
&= \sum_{v_s=0}^m \sum_{v_b=0}^m \left(t_{v_s, v_b} \cdot f_b(v_b) - v_s \cdot q_{v_s, v_b} \cdot f_b(v_b) \right) \cdot f_s(v_s) + \sum_{v_b=0}^m \sum_{v_s=0}^m \left(v_b \cdot q_{v_s, v_b} \cdot f_s(v_s) - t_{v_s, v_b} \cdot f_s(v_s) \right) \cdot f_b(v_b) \\
&= \sum_{v_s=0}^m U_s(v_s) \cdot f_s(v_s) + \sum_{v_b=0}^m U_b(v_b) \cdot f_b(v_b) \\
&\geq \sum_{v_s=0}^m \left(U_s(m) + \sum_{v_i=v_s+1}^m \bar{q}_s(v_i) \right) \cdot f_s(v_s) + \sum_{v_b=0}^m \left(U_b(0) + \sum_{v_j=0}^{v_b-1} \bar{q}_b(v_j) \right) \cdot f_b(v_b) \\
&= U_s(m) + \sum_{v_s=0}^m \sum_{v_i=v_s+1}^m \bar{q}_s(v_i) \cdot f_s(v_s) + U_b(0) + \sum_{v_b=0}^m \sum_{v_j=0}^{v_b-1} \bar{q}_b(v_j) \cdot f_b(v_b) \\
&= U_s(m) + U_b(0) + \sum_{v_s=0}^m \bar{q}_s(v_s) \cdot F_s(v_s - 1) + \sum_{v_b=0}^m \bar{q}_b(v_b) \cdot (1 - F_b(v_b)) \\
&= U_s(m) + U_b(0) + \sum_{v_s=0}^m \sum_{v_b=0}^m \left(F_s(v_s - 1) \cdot f_b(v_b) + (1 - F_b(v_b)) \cdot f_s(v_s) \right) \cdot q_{v_s, v_b},
\end{aligned}$$

where the inequality uses Lemma A.1 in the Appendix by setting $r_s = m$ and $r_b = 0$. Rearranging the first and last term of these expressions leads to the desired inequality. In a similar way of rearranging and using the other inequalities of Lemma A.1, we obtain the left inequality of this lemma. \square

The following properties hold for an EPIR mechanism.

Proposition A.2. *Let the direct mechanism (q, t^+, t^-) be EPIR. Then every t defined by (t^+, t^-) satisfies:*

- (a) $0 \leq t_{v_s, v_b} \leq m$
- (b) if $v_s > v_b$, then $q_{v_s, v_b} = 0$
- (c) if $q_{v_s, v_b} = 0$, then $t_{v_s, v_b} = 0$
- (d) $\bar{t}_s(m) = m \cdot \bar{q}_s(m)$
- (e) $\bar{t}_b(0) = 0$.

Proof. By the definition of EPIR, we have that for all possible valuations v_s and v_b

$$v_s \cdot q_{v_s, v_b} \leq t_{v_s, v_b} \leq v_b \cdot q_{v_s, v_b}. \quad (11)$$

Observe that (11) directly implies (a), (b) and (c).

In order to prove (d), see that (11) assures that $t_{m, m} = m \cdot q_{m, m}$. Also notice from property (b) that $q_{m, 0} = \dots = q_{m, m-1} = 0$. Hence $\bar{q}_s(m) = q_{m, m}$ and by (c) it holds that $\bar{t}_s(m) = t_{m, m}$. In conclusion, we find that indeed $\bar{t}_s(m) = m \cdot \bar{q}_s(m)$.

As last for (e), conclude from (11) that $t_{0,0} = 0$ and notice from (b) that $q_{1,0} = \dots = q_{m,0} = 0$. So by (c) it holds that $\bar{t}_b(0) = t_{0,0}$ and hence $\bar{t}_b(0) = 0$. \square

Proof of Theorem 5.1. Let us first proof the “only if” part. Assume q is IIR-implementable. By Lemma 3.4, we know that $\bar{q}_s(\cdot)$ is weakly decreasing and $\bar{q}_b(\cdot)$ is weakly increasing. For every (q, t^+, t^-) satisfying IC and IIR, we know by Lemma 3.6 that $U_s(m) + U_b(0) \leq u_b - l_s$ and we know that IIR assures that $U_s(m) \geq 0$ and $U_b(0) \geq 0$. Combined this means that inequality (3) needs to hold.

In order to complete the “if part” of the proof, suppose that $\bar{q}_s(\cdot)$ is weakly decreasing, $\bar{q}_b(\cdot)$ is weakly increasing and q satisfies (3). We will construct a t such that for every (t^+, t^-) defining this t , we have that (q, t^+, t^-) satisfies IC and IIR:

$$\begin{aligned} t_{v_s, v_b} &= \sum_{v_j=1}^{v_b} v_j \cdot \left(\bar{q}_b(v_j) - \bar{q}_b(v_j - 1) \right) - \sum_{v_i=0}^{v_s-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) \\ &\quad + \sum_{v_i=0}^{m-1} v_i \cdot \left(1 - F_s(v_i) \right) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right). \end{aligned}$$

Let us show that (q, t^+, t^-) indeed satisfies IC and IIR. Notice that all the three sums on the right-hand side are nonnegative, due to our assumptions on $\bar{q}_s(\cdot)$ and $\bar{q}_b(\cdot)$. The first term is chosen so that (q, t^+, t^-) satisfies incentive compatibility for the buyer, the second term is chosen to satisfy incentive compatibility for the seller and the last term is chosen in such a way that (q, t^+, t^-) satisfies $U_b(0) = 0$ and $U_s(m) \geq 0$. Once we prove these properties, it will follow from Lemma 3.4 that (q, t^+, t^-) also satisfies IIR.

To check incentive compatibility for the seller, observe that for types $r_s > v_s$

$$\begin{aligned} &U_s(v_s) - U_s(r_s, v_s) \\ &= \bar{t}_s(v_s) - v_s \cdot \bar{q}_s(v_s) - \left(\bar{t}_s(r_s) - v_s \cdot \bar{q}_s(r_s) \right) \\ &= \bar{t}_s(v_s) - \bar{t}_s(r_s) - v_s \cdot \left(\bar{q}_s(v_s) - \bar{q}_s(r_s) \right) \\ &= \sum_{v_b=0}^m \sum_{v_i=0}^{r_s-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) \cdot f_b(v_b) - \sum_{v_b=0}^m \sum_{v_i=0}^{v_s-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) \cdot f_b(v_b) \\ &\quad - v_s \cdot \left(\bar{q}_s(v_s) - \bar{q}_s(r_s) \right) \\ &= \sum_{v_i=v_s}^{r_s-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) - \sum_{v_i=v_s}^{r_s-1} v_s \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) \\ &\geq 0 \end{aligned}$$

and for types $r_s < v_s$

$$\begin{aligned} U_s(v_s) - U_s(r_s, v_s) &= - \sum_{v_i=r_s}^{v_s-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) + \sum_{v_i=r_s}^{v_s-1} v_s \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i + 1) \right) \\ &\geq 0. \end{aligned}$$

A similar analysis can be conducted to prove incentive compatibility for the buyer. To check IIR, note that

$$\begin{aligned}
U_b(0) &= - \sum_{v_s=0}^m t_{v_s,0} \cdot f_s(v_s) \\
&= \sum_{v_s=0}^m \sum_{v_i=0}^{v_s-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) \cdot f_s(v_s) - \sum_{v_s=0}^m \sum_{v_i=0}^{m-1} v_i \cdot \left(1 - F_s(v_i) \right) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) \cdot f_s(v_s) \\
&= \sum_{v_s=1}^m \sum_{v_i=0}^{v_s-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) \cdot f_s(v_s) - \sum_{v_i=0}^{m-1} v_i \cdot \left(1 - F_s(v_i) \right) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) \\
&= \sum_{v_i=0}^{m-1} v_i \cdot \left(1 - F_s(v_i) \right) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) - \sum_{v_i=0}^{m-1} v_i \cdot \left(1 - F_s(v_i) \right) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
U_s(m) &= \sum_{v_b=0}^m t_{m,v_b} \cdot f_b(v_b) - m \cdot \bar{q}_s(m) \\
&= \sum_{v_b=0}^m \left(\sum_{v_j=1}^{v_b} v_j \cdot \left(\bar{q}_b(v_j) - \bar{q}_b(v_j-1) \right) - \sum_{v_i=0}^{m-1} v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) \right) \cdot f_b(v_b) - m \cdot \bar{q}_s(m) \\
&= \sum_{v_b=1}^m \sum_{v_j=1}^{v_b} v_j \cdot \left(\bar{q}_b(v_j) - \bar{q}_b(v_j-1) \right) \cdot f_b(v_b) - \sum_{v_i=0}^{m-1} v_i \cdot F_s(v_i) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) - m \cdot \bar{q}_s(m) \\
&= u_b - \sum_{v_s=0}^{m-1} \left(v_s \cdot \bar{q}_s(v_s) + \sum_{v_i=v_s+1}^m \bar{q}_s(v_i) \right) \cdot f_s(v_s) - m \cdot \bar{q}_s(m) \cdot f_s(m) \\
&= u_b - \sum_{v_s=0}^{m-1} \sum_{v_i=v_s}^{m-1} \left(v_i \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) + m \cdot \bar{q}_s(m) \right) \cdot f_s(v_s) - m \cdot \bar{q}_s(m) \cdot f_s(m) \\
&= u_b - l_s \\
&\geq 0.
\end{aligned}$$

□

Proof of Theorem 6.2. Let us show that (q, t^+, t^-) satisfies IIR and WIC for the seller. What should be noted is that for each type $v_s < m$, we have that

$$\bar{t}_s(v_s) = \sum_{v_i=v_s}^{m-1} (v_i + \Delta_s) \cdot \left(\bar{q}_s(v_i) - \bar{q}_s(v_i+1) \right) + m \cdot \bar{q}_s(m) \geq 0.$$

Also for each v_s and $v_s + 1$, we have that

$$\bar{t}_s(v_s) - \bar{t}_s(v_s + 1) = (v_s + \Delta_s) \left(\bar{q}_s(v_s) - \bar{q}_s(v_s + 1) \right).$$

Since $0 \leq \Delta_s \leq 1$, the mechanism (q, t^+, t^-) is WIC and IIR for the seller.

It only remains to show that (q, t^+, t^-) satisfies IIR and IC for the buyer. For each type of buyer $v_b > 0$, we have that

$$\bar{t}_b(v_b) = \sum_{v_i=1}^{v_b} (v_i - 1 + \Delta_b) \cdot \left(\bar{q}_b(v_i) - \bar{q}_b(v_i - 1) \right) \geq 0.$$

Then for each type of buyer v_b and $\frac{v_b-1}{m}$, it holds that

$$\bar{t}_b(v_b) - \bar{t}_b(v_b - 1) = (v_b - 1 + \Delta_b) \left(\bar{q}_b(v_b) - \bar{q}_b(v_b - 1) \right).$$

Since $0 \leq \Delta_b \leq 1$, the mechanism is WIC and IIR for the buyer. As the mechanism satisfies WIC for both the seller and the buyer, according to Lemma 3.3 it also satisfies IC. Hence we have shown that the above payment matrix assures that the mechanism is IIR, IC and has only nonzero prices if there is trade. \square

Proof of Lemma 6.3. Let (t^{T+}, t^{T-}) denote the transfer for the transposed trading rule. Define $t^{T-} = 0$ and

$$t_{v_s, v_b}^{T+} = \begin{cases} m - t_{m-v_b, m-v_s} & \text{if } q_{m-v_b, m-v_s} = 1. \\ 0 & \text{if } q_{m-v_b, m-v_s} = 0. \end{cases}$$

We show that if (q, t^+, t^-) satisfies IC and EPIR, then (q^T, t^{T+}, t^{T-}) also satisfies IC and EPIR. Assume that (q, t^+, t^-) satisfies IC and EPIR. By construction, (q^T, t^{T+}, t^{T-}) is EPIR. Thus it remains to show that (q^T, t^{T+}, t^{T-}) is WIC and therefore IC. Let $U_s^T(v_s)$ denote the expected utility of the seller in the transposed mechanism. We show that $U_s^T(v_s) \geq U_s^T(v_s + 1, v_s)$ for every $v_s = 0, \dots, m-1$. The inequality for the other deviation direction and the proof for the buyer can be conducted in a similar fashion. First, observe that for every v_s :

$$\bar{q}_s^T(v_s) = \bar{q}_b(m - v_s).$$

Then, notice that

$$\begin{aligned} \bar{t}_s^T(v_s) &= \sum_{v_b=0}^m t_{v_s, v_b}^{T+} \cdot \frac{1}{m+1} \\ &= \sum_{v_b=0}^m (m - t_{m-v_b, m-v_s}) \cdot q_{m-v_b, m-v_s} \cdot \frac{1}{m+1} \\ &= m \cdot \bar{q}_b(m - v_s) - \bar{t}_b(m - v_s) \end{aligned}$$

and thus

$$\begin{aligned}
U_s^T(v_s) &= \bar{t}_s^T(v_s) - v_s \cdot \bar{q}_s^T(v_s) \\
&= (m - v_s) \cdot \bar{q}_b(m - v_s) - \bar{t}_b(m - v_s) \\
&= U_b(m - v_s) \\
&\geq U_b(m - (v_s + 1), m - v_s) \\
&= (m - v_s) \cdot \bar{q}_b(m - (v_s + 1)) - \bar{t}_b(m - (v_s + 1)) \\
&= \bar{t}_s^T(v_s + 1) - v_s \cdot \bar{q}_s^T(v_s + 1) \\
&= U_s^T(v_s + 1, v_s).
\end{aligned}$$

□

The following lemma presents necessary and sufficient conditions for corner mechanisms to be EPIR-implementable.

Lemma A.3. *Assume $f_i(v_i) = \frac{1}{m+1}$ for all valuations v_i for $i = s, b$ and let q be a corner matrix with $k > 0$ and $\ell > 0$. Then q is EPIR-implementable if and only if there exist $x, y, z \in \mathbb{R}$ satisfying*

$$k \cdot (m - 1) \leq z + k \cdot x - y \leq k \cdot m \quad (12)$$

$$0 \leq z + \ell \cdot y - x \leq \ell \quad (13)$$

$$k \leq x \leq k + 1 - q_{m,m} \quad (14)$$

$$m - \ell - 1 + q_{0,0} \leq y \leq m - \ell \quad (15)$$

$$0 \leq z \leq m. \quad (16)$$

Proof. To prove the "only if" part, assume that q is EPIR-implementable. WIC for the seller implies that $0 \leq \sum_{v_b=m-\ell}^m t_{0,v_b} - t_{1,m} \leq \ell$ and that $t_{1,m} = t_{2,m} = \dots = t_{k,m}$. If $k < m$, then WIC implies that $k \leq t_{k,m} \leq k + 1$ or else if $k = m$ EPIR implies that $t_{m,m} = m$. In a similar way of reasoning we can conclude that $k \cdot (m - 1) \leq \sum_{v_s=0}^k t_{v_s,m} - t_{0,m-1} \leq k \cdot m$, $t_{0,m-1} = \dots = t_{0,m-\ell+1} = t_{0,m-\ell}$ and that $m - \ell - 1 \leq t_{0,m-\ell} \leq m - \ell$ if $0 \leq \ell < m$ or $t_{0,0} = 0$ if $\ell = m$. If we then let $x = t_{1,m}$, $y = t_{0,m-1}$ and $z = t_{0,m}$, we end up with the inequalities mentioned in the Lemma.

For the "if part" assume there exists a x, y and z satisfying the inequalities. Set $t_{1,m} = \dots = t_{k,m} = x$, $t_{0,m-\ell} = \dots = t_{0,m-1} = y$ and $t_{0,m} = z$. Clearly, due to the constraints on x, y and z these prices will satisfy WIC and thus IC. Furthermore, since $0 \leq k \leq x \leq k + 1 - q_{m,m} \leq m$, $0 \leq m - \ell - 1 + q_{0,0} \leq y \leq m - \ell \leq m$ and $0 \leq z \leq m$ these prices will also satisfy EPIR. □

Proof of Theorem 7.3. We want to find a q that maximizes

$$\sum_{v_s=0}^m \sum_{v_b=0}^m (v_b - v_s) \cdot q_{v_s,v_b} \cdot f_s(v_s) \cdot f_b(v_b)$$

subject to constraint (3): $l_s \leq u_b$. Setting up the Lagrangian:

$$\begin{aligned}
\mathcal{L}(v_s, v_b, \lambda) &= \sum_{v_s=0}^m \sum_{v_b=0}^m \left(v_b + \lambda \cdot \left(v_b - \frac{1 - F_b(v_b)}{f_b(v_b)} \right) - v_s - \lambda \cdot \left(v_s + \frac{F_s(v_s - 1)}{f_s(v_s)} \right) \right) \cdot q_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b) \\
&= (1 + \lambda) \cdot \sum_{v_s=0}^m \sum_{v_b=0}^m \left(v_b - \frac{\lambda}{1 + \lambda} \frac{1 - F_b(v_b)}{f_b(v_b)} - v_s - \frac{\lambda}{1 + \lambda} \frac{F_s(v_s - 1)}{f_s(v_s)} \right) \cdot q_{v_s, v_b} \cdot f_s(v_s) \cdot f_b(v_b).
\end{aligned}$$

This Lagrangian is maximized by q^α with $\alpha = \frac{\lambda}{1+\lambda}$. We know that for q^α , $\bar{q}_s(\cdot)$ is weakly decreasing and $\bar{q}_b(\cdot)$ is weakly increasing. Furthermore, since $U_s(m) = U_b(0) = 0$ we know due to Lemma 3.6 that $l_s \leq u_b$ and thus we can conclude from Theorem 5.1 that q^α is IIR-implementable. \square

References

- Bulow, J., Roberts, J. 1989. The simple economics of optimal auctions. *The Journal of Political Economy* 97, 1060-1090.
- Chatterjee, K., Samuelson, W. 1983. Bargaining under incomplete information. *Operations Research* 31, 835-851.
- Gresik, T. 1991a. Ex ante efficient, ex post individually rational trade. *Journal of Economic Theory* 53, 131-145.
- Gresik, T. 1991b. Efficient bilateral trade with statistically dependent beliefs. *Journal of Economic Theory* 53, 199-205.
- Gresik, T., Satterthwaite, M. 1983. The number of traders required to make a market competitive: the beginnings of a theory. Discussion Papers 551, Northwestern University, Center for Mathematical Studies in Economics and Management Science.
- Hagerty, K., Rogerson, W. 1987. Robust trading mechanisms. *Journal of Economic Theory* 42, 94-107.
- Matsuo, T. 1989. On incentive compatible, individually rational, and ex post efficient mechanisms for bilateral trading. *Journal of Economic Theory* 49, 189-194.
- Myerson, R., Satterthwaite, M. 1983. Efficient mechanisms for bilateral trading. *Journal of Economic Theory* 29, 265-281.
- Vohra, R. 2011. Mechanism design: a linear programming approach. Cambridge University Press, New York.